AN INTRODUCTION TO HARMONIC ANALYSIS

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Third Corrected Edition

Preface

Harmonic analysis is the study of objects (functions, measures, etc.), defined on topological groups. The group structure enters into the study by allowing the consideration of the translates of the object under study, that is, by placing the object in a translation-invariant space. The study consists of two steps. First: finding the "elementary components" of the object, that is, objects of the same or similar class, which exhibit the simplest behavior under translation and which "belong" to the object under study (harmonic or spectral *analysis*); and second: finding a way in which the object can be construed as a combination of its elementary components (harmonic or spectral *synthesis*).

The vagueness of this description is due not only to the limitation of the author but also to the vastness of its scope. In trying to make it clearer, one can proceed in various ways[†]; we have chosen here to sacrifice generality for the sake of concreteness. We start with the circle group $\mathbb{T}$ and deal with classical Fourier series in the first five chapters, turning then to the real line in Chapter VI and coming to locally compact abelian groups, only for a brief sketch, in Chapter VII. The philosophy behind the choice of this approach is that it makes it easier for students to grasp the main ideas and gives them a large class of concrete examples which are essential for the proper understanding of the theory in the general context of topological groups. The presentation of Fourier series and integrals differs from that in [1], [7], [8], and [28] in being, I believe, more explicitly aimed at the general (locally compact abelian) case.

The last chapter is an introduction to the theory of commutative Banach algebras. It is biased, studying Banach algebras mainly as a tool in harmonic analysis.

This book is an expanded version of a set of lecture notes written

[†]Hence the indefinite article in the title of the book.

for a course which I taught at Stanford University during the spring and summer quarters of 1965. The course was intended for graduate students who had already had two quarters of the basic "real-variable" course. The book is on the same level: the reader is assumed to be familiar with the basic notions and facts of Lebesgue integration, the most elementary facts concerning Borel measures, some basic facts about holomorphic functions of one complex variable, and some elements of functional analysis, namely: the notions of a Banach space, continuous linear functionals, and the three key theorems—"the closed graph", the Hahn-Banach, and the "uniform boundedhess" theorems. All the prerequisites can be found in [23] and (except, for the complex variable) in [22]. Assuming these prerequisites, the book, or most of it, can be covered in a one-year course. A slower moving course or one shorter than a year may exclude some of the starred sections (or subsections). Aiming for a one-year course forced the omission not only of the more general setup (non-abelian groups are not even mentioned), but also of many concrete topics such as Fourier analysis on $\mathbb{R}^n, n > l$, and finer problems of harmonic analysis in $\mathbb{T}$ or $\mathbb{R}$ (some of which can be found in [13]). Also, some important material was cut into exercises, and we urge the reader to do as many of them as he can.

The bibliography consists mainly of books, and it is through the bibliographies included in these books that the reader is to become familiar with the many research papers written on harmonic analysis. Only some, more recent, papers are included in our bibliography. In general we credit authors only seldom—most often for identification purposes. With the growing mobility of mathematicians, and the happy amount of oral communication, many results develop within the mathematical folklore and when they find their way into print it is not always easy to determine who deserves the credit. When I was writing Chapter Ill of this book, I was very pleased to produce the simple elegant proof of Theorem 1.6 there. I could swear I did it myself until I remembered two days later that six months earlier, "over a cup of coffee," Lennart Carleson indicated to me this same proof.

The book is divided into chapters, sections, and subsections. The chapter numbers are denoted by roman numerals and the sections and subsections, as well as the exercises, by arabic numerals. In cross references within the same chapter, the chapter number is omitted; thus Theorem llI.1.6, which is the theorem in subsection 6 of Section 1 of Chapter Ill, is referred to as Theorem 1.6 within Chapter IlI, and

Theorem Ill.1.6 elsewhere. The exercises are gathered at the end of the sections, and exercise V.1.1 is the first exercise at the end of Section 1, Chapter V. Again, the chapter number is omitted when an exercise is referred to within the same chapter. The ends of proofs are marked by a triangle (◀).

The book was written while I was visiting the University of Paris and Stanford University and it owes its existence to the moral and technical help 1 was so generously given in both places. During the writing I have benefitted from the advice and criticism of many friends; 1 would like to thank them all here. Particular thanks are due to L. Carleson, K. DeLeeuw, J.-P. Kahane, O.C. McGehee, and W. Rudin. I would also like to thank the publisher for the friendly cooperation in the production of this book.

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The 2002 edition

The second edition was essentially identical with the first, except for the correction of a few misprints. The current edition has some more misprints and "miswritings" corrected, and some material added: an additional section in the first chapter, a few exercises, and an additional appendix. The added material does not reflect the progress in the field in the past thirty or forty years. Almost all of it could, and should have been included in the first edition of the book.

Stanford
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AN INTRODUCTION TO

HARMONIC ANALYSIS

Chapter I

Fourier Series on $\mathbb{T}$

We denote by $\mathbb{R}$ the additive group of real numbers and by $\mathbb{Z}$ the subgroup consisting of the integers. The group $\mathbb{T}$ is defined as the quotient $\mathbb{R}/2\pi\mathbb{Z}$ where, as indicated by the notation, $2\pi\mathbb{Z}$ is the group of the integral multiples of 2π. There is an obvious identification between functions on $\mathbb{T}$ and 2π-periodic functions on $\mathbb{R}$, which allows an implicit introduction of notions such as continuity, differentiability, etc. for functions on $\mathbb{T}$. The *Lebesgue measure* on $\mathbb{T}$, also, can be defined by means of the preceding identification: a function f is integrable on $\mathbb{T}$ if the corresponding 2π-periodic function, which we denote again by f, is integrable on $[0, 2\pi)$ and we set

$$\int_{\mathbb{T}} f(t)dt = \int_0^{2\pi} f(x)dx.$$

In other words, we consider the interval $[0, 2\pi)$ as a model for $\mathbb{T}$ and the Lebesgue measure dt on $\mathbb{T}$ is the restriction of the Lebesgue measure of $\mathbb{R}$ to $[0, 2\pi)$. The total mass of dt on $\mathbb{T}$ is equal to 2π and many of our formulas would be simpler if we normalized dt to have total mass 1, that is, if we replace it by $dx/2\pi$. Taking intervals on $\mathbb{R}$ as "models" for $\mathbb{T}$ is very convenient, however, and we choose to put $dt = dx$ in order to avoid confusion. We "pay" by having to write the factor $1/2\pi$ in front of every integral.

An all-important property of dt on $\mathbb{T}$ is its translation invariance, that is, for all $t_0 \in \mathbb{T}$ and f defined on $\mathbb{T}$,

$$\int f(t - t_0)dt = \int f(t)dt^{\dagger}$$

†Throughout this chapter, integrals with unspecified limits of integration are taken over $\mathbb{T}$.

1 FOURIER COEFFICIENTS

1.1 We denote by $L^1(\mathbb{T})$ the space of all (equivalence† classes of) complex-valued, Lebesgue integrable functions on $\mathbb{T}$. For $f \in L^1(\mathbb{T})$ we put

$$\|f\|_{L^1} = \frac{1}{2\pi}\int_{\mathbb{T}} |f(t)|dt.$$

It is well known that $L^1(\mathbb{T})$, with the norm so defined, is a Banach space.

DEFINITION: A *trigonometric polynomial* on $\mathbb{T}$ is an expression of the form

$$P \sim \sum_{n=-N}^{N} a_n e^{int}. \tag{1.1}$$

The numbers n appearing in (1.1) are called the frequencies of P; the largest integer n such that $|a_n| + |a_{-n}| \neq 0$ is called *the degree of* P. The values assumed by the index n are integers so that each of the summands in (1.1) is a function on $\mathbb{T}$. Since (1.1) is a finite sum, it represents a function, which we denote again by P, defined for each $t \in \mathbb{T}$ by

$$P(t) = \sum_{n=-N}^{N} a_n e^{int}. \tag{1.2}$$

Let P be defined by (1.2). Knowing the function P we can compute the coefficients a_n by the formula

$$a_n = \frac{1}{2\pi}\int P(t)e^{-int}dt \tag{1.3}$$

which follows immediately from the fact that for integers j,

$$\frac{1}{2\pi}\int e^{ijt}dt = \begin{cases} 1 & \text{if } j = 0, \\ 0 & \text{if } j \neq 0. \end{cases}$$

Thus we see that the function P determines the expression (1.1) and there seems to be no point in keeping the distinction between the expression (1.1) and the function P; we shall consider trigonometric polynomials as both formal expressions and functions.

† $f \sim g$ if $f(t) = g(t)$ almost everywhere

1.2 DEFINITION: *A trigonometric series on* $\mathbb{T}$ is an expression of the form

$$S \sim \sum_{n=-\infty}^{\infty} a_n e^{int}. \tag{1.4}$$

Again, n assumes integral values; however, the number of terms in (1.4) may be infinite and there is no assumption whatsoever about the size of the coefficients or about convergence. The conjugate‡ of the series (1.4) is, by definition, the series

$$\tilde{S} \sim \sum_{n=-\infty}^{\infty} -i \operatorname{sgn}(n) a_n e^{int}.$$

where $\operatorname{sgn}(n) = 0$ if $n = 0$ and $\operatorname{sgn}(n) = n/|n|$ otherwise.

1.3 Let $f \in L^1(\mathbb{T})$. Motivated by (1.3) we define the *n*th Fourier coefficient of f by

$$\hat{f}(n) = \frac{1}{2\pi} \int f(t) e^{-int} dt. \tag{1.5}$$

DEFINITION: *The Fourier series* $S[f]$ *of a function* $f \in L^1(\mathbb{T})$ is the trigonometric series

$$S[f] \sim \sum_{-\infty}^{\infty} \hat{f}(n) e^{int}.$$

The series conjugate to $S[f]$ will be denoted by $\widetilde{S}[f]$ and referred to as the conjugate Fourier series of f. We shall say that a trigonometric series is a Fourier series if it is the Fourier series of some $f \in L^1(\mathbb{T})$.

1.4 We turn to some elementary properties of Fourier coefficients.

Theorem. *Let* $f, g \in L^1(\mathbb{T})$*, then*

(a) $(\widehat{f+g})(n) = \hat{f}(n) + \hat{g}(n)$.

(b) For any complex number α

$$(\widehat{\alpha f})(n) = \alpha \hat{f}(n).$$

(c) If $\bar{f}$ *is the complex conjugate*§ *of* f *then* $\hat{\bar{f}}(n) = \overline{\hat{f}(-n)}$.

‡See Chapter III for motivation of the terminology.

§Defined by: $\bar{f}(t) = \overline{f(t)}$) for all $t \in \mathbb{T}$.

(d) Denote $f_\tau(t) = f(t-\tau), \quad \tau \in \mathbb{T}$*; then*

$$\hat{f_\tau}(n) = \hat{f}(n)e^{-in\tau}.$$

(e) $|\hat{f}(n)| \le \frac{1}{2\pi}\int |f(t)|dt = \|f\|_{L^1}$

The proofs of (a) through (e) follow immediately from (1.5) and the details are left to the reader.

1.5 Corollary. *Assume* $f_j \in L^1(\mathbb{T})$, $j = 0, 1, \ldots,$ *and* $\|f_j - f_0\|_{L^1} \to 0$. *Then* $\hat{f}(n) \to \hat{f_0}(n)$ *uniformly.*

1.6 Theorem. *Let* $f \in L^1(\mathbb{T})$*, assume* $\hat{f}(0) = 0$*, and define*

$$F(t) = \int_0^t f(\tau)d\tau.$$

Then F *is continuous,* 2π*-periodic, and*

(1.6) $$\hat{F}(n) = \frac{1}{in}\hat{f}(n), \qquad n \ne 0.$$

PROOF: The continuity (and, in fact, the absolute continuity) of F is evident. The periodicity follows from

$$F(t+2\pi) - F(t) = \int_t^{t+2\pi} f(\tau)d\tau = 2\pi\hat{f}(0) = 0,$$

and (1.6) is obtained through integration by parts:

$$\hat{F}(n) = \frac{1}{2\pi}\int_0^{2\pi} F(t)e^{-int}dt = \frac{-1}{2\pi}\int_0^{2\pi} F'(t)\frac{1}{-in}e^{-int}dt = \frac{1}{in}\hat{f}. \quad ◀$$

1.7 We now define the convolution operation in $L^1(\mathbb{T})$. The reader will notice the use of the group structure of $\mathbb{T}$ and of the invariance of dt in the subsequent proofs.

Theorem. *Let* $f, g \in L^1(\mathbb{T})$*. For almost all* t*, the function* $f(t-\tau)g(\tau)$ *is integrable (as a function of* τ *on* $\mathbb{T}$*), and, if we write*

(1.7) $$h(t) = \frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau,$$

then $h \in L^1(\mathbb{T})$ *and*

(1.8) $$\|h\|_{L^1} \le \|f\|_{L^1}\|g\|_{L^1}.$$

Moreover

(1.9) $$\hat{h}(n) = \hat{f}(n)\hat{g}(n) \quad \textit{for all } n.$$

PROOF: The functions $f(t-\tau)$ and $g(\tau)$, considered as functions of the two variables (t, x), are clearly measurable, hence so is

$$F(t,\tau) = f(t-\tau)g(\tau).$$

For every τ, $F(t,\tau)$ is just a constant multiple of f_τ, hence integrable dt, and

$$\frac{1}{2\pi}\int\left(\frac{1}{2\pi}\int|F(t,\tau)|dt\right)d\tau = \frac{1}{2\pi}\int|g(\tau)|\cdot\|f\|_{L^1}d\tau = \|f\|_{L^1}\|g\|_{L^1}$$

Hence, by the theorem of Fubini, $f(t-\tau)g(\tau)$ is integrable (over $(0, 2\pi)$) as a function of τ for almost all t, and

$$\begin{aligned}\frac{1}{2\pi}\int|h(t)|dt = \frac{1}{2\pi}\int\left|\frac{1}{2\pi}\int F(t,\tau)d\tau\right|dt &\leq \frac{1}{4\pi^2}\iint|F(t,\tau)|dt\,d\tau \\ &= \|f\|_{L^1}\|g\|_{L^1}\end{aligned}$$

which establishes (1.8). In order to prove (1.9) we write

$$\begin{aligned}\hat{h}(n) =& \frac{1}{2\pi}\int h(t)e^{-int}dt = \frac{1}{4\pi^2}\iint f(t-\tau)e^{-in(t-\tau)}g(\tau)e^{-in\tau}dt\,d\tau \\ =& \frac{1}{2\pi}\int f(t)e^{-int}dt\cdot\frac{1}{2\pi}\int g(\tau)e^{-in\tau}d\tau = \hat{f}(n)\hat{g}(n).\end{aligned}$$

As above the change in the order of integration is justified by Fubini's theorem. ◀

1.8 DEFINITION: The *convolution* $f * g$ of the ($L^1(\mathbb{T})$ functions) f and g is the function h defined by (1.8). Using the star notation for the convolution, we can write (1.9):

$$\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n). \tag{1.10}$$

Theorem. *The convolution operation in $L^1(\mathbb{T})$ is commutative, associative, and distributive (with respect to the addition).*

PROOF: The change of variable $\vartheta = t - \tau$ gives

$$\frac{1}{2\pi}\int f(t-\tau)g(\tau)d\tau = \frac{1}{2\pi}\int g(t-\vartheta)f(\vartheta)d\vartheta,$$

that is,

$$f * g = g * f.$$

If $f_1, f_2, f_3 \in L^1(\mathbb{T})$, then

$$[(f_1 * f_2) * f_3](t) = \frac{1}{4\pi^2} \iint f_1(t-u-\tau) f_2(u) f_3(\tau) du\, d\tau =$$
$$\frac{1}{4\pi^2} \iint f_1(t-\omega) f_2(\omega-\tau) f_3(\tau) d\omega\, d\tau = [f_1 * (f_2 * f_3)](t).$$

Finally, the distributive law

$$f_1 * (f_2 + f_3) = f_1 * f_2 + f_1 * f_3$$

is evident from (1.7). ◀

1.9 Lemma. *Assume $f \in L^1(\mathbb{T})$ and let $\varphi(t) = e^{int}$ for some integer n. Then*

$$(\varphi * f)(t) = \hat{f}(n) e^{int}.$$

PROOF:

$$(\varphi * f)(t) = \frac{1}{2\pi} \int e^{in(t-\tau)} f(\tau) d\tau = e^{int} \frac{1}{2\pi} \int \frac{1}{2\pi} \int f(\tau) e^{-in\tau} d\tau.$$ ◀

Corollary. *If $f \in L^1(\mathbb{T})$ and $k(t) = \sum_{-N}^{N} a_n e^{int}$, then*

(1.11) $$(k * f)(t) = \sum_{-N}^{N} a_n \hat{f}(n) e^{int}.$$

EXERCISES FOR SECTION 1

1. Compute the Fourier coefficients of the following functions (defined by their values on $[-\pi, \pi)$):

(a) $$f(t) = \begin{cases} \sqrt{2\pi} & |t| < \frac{1}{2} \\ 0 & \frac{1}{2} \le |t| \le \pi. \end{cases}$$

(b) $$\Delta(t) = \begin{cases} 1 - |t| & |t| < 1 \\ 0 & 1 \le |t| \le \pi. \end{cases}$$

What relation do you see between f and Δ ?

(c) $$g(t) = \begin{cases} 1 & -1 < t \le 0 \\ -1 & 0 < t < 1 \\ 0 & 1 \le |t|, \end{cases}$$

What relation do you see between g and Δ?

(d) $$h(t) = t \qquad -\pi < t < \pi.$$

2. Remembering Euler's formulas

$$\cos t = \frac{1}{2}(e^{it} + e^{-it}), \qquad \sin t = \frac{1}{2i}(e^{it} - e^{-it}),$$

or

$$e^{it} = \cos t + i \sin t\,,$$

show that the Fourier series of a function $f \in L^1(\mathbb{T})$ is formally equal to

$$\frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n \cos nt + B_n \sin nt)$$

where $A_n = \hat{f}(n) + \hat{f}(-n)$ and $B_n = i(\hat{f}(n) - \hat{f}(-n))$. Equivalently:

$$A_n = \frac{1}{\pi}\int f(t)\cos nt\, dt$$

$$B_n = \frac{1}{\pi}\int f(t)\sin nt\, dt.$$

Show also that if f is real valued, then A_n and B_n are all real; if f is even, that is, if $f(t) = f(-t)$, then $B_n = 0$ for all n; and if f is odd, that is, if $f(t) = -f(-t)$, then $A_n = 0$ for all n.

3. Show that if $S \sim \sum a_j \cos jt$, then $\tilde{S} \sim \sum a_j \sin jt$.

4. Let $f \in L^1(\mathbb{T})$ and let $P(t) = \sum_{-N}^{N} a_n e^{int}$. Compute the Fourier coefficients of the function fP.

5. Let $f \in L^1(\mathbb{T})$, let m be a positive integer, and write

$$f_{(m)}(t) = f(mt).$$

Show

$$\widehat{f_{(m)}}(n) = \begin{cases} \hat{f}(\frac{n}{m}) & \text{if } m \mid n. \\ 0 & \text{if } m \nmid n \end{cases}$$

6. The trigonometric polynomial $\cos nt = \frac{1}{2}(e^{int} + e^{-int})$ is of degree n and has $2n$ zeros on $\mathbb{T}$. Show that no trigonometric polynomial of degree $n > 0$ can have more than $2n$ zeros on $\mathbb{T}$.

Hint: Identify $\sum_{-n}^{n} a_j e^{ijt}$ on $\mathbb{T}$ with $z^{-n}\sum_{-n}^{n} a_j z^{n+j}$ on $|z| = 1$.

7. Denote by C^* the multiplicative group of complex numbers different from zero. Denote by T^* the subgroup of all $z \in C^*$ such that $|z| = 1$. Prove that if G is a subgroup of C^* which is compact (as a set of complex numbers), then $G \subseteq T^*$.

8. Let G be a compact proper subgroup of $\mathbb{T}$. Prove that G is finite and determine its structure.

Hint: Show that G is discrete.

9. Let G be an infinite subgroup of $\mathbb{T}$. Prove that G is dense in $\mathbb{T}$.

Hint: The closure of G in $\mathbb{T}$ is a compact subgroup.

10. Let α be an irrational multiple of 2π. Prove that $\{n\alpha \pmod{2\pi}\}_{n\in\mathbb{Z}}$ is dense in $\mathbb{T}$.

11. Prove that a continuous homomorphism of $\mathbb{T}$ into C^* is necessarily given by an exponential function.

Hint: Use exercise 7 to show that the mapping is into T^*; determine the mapping on "small" rational multiples of 2π and use exercise 9.

12 If E is a subset of $\mathbb{T}$ and $\tau_0 \in \mathbb{T}$, we define $E + \tau_0 = \{t + \tau_0 : t \in E\}$; we say that E is invariant under translation by τ if $E = E + \tau$. Show that, given a set E, the set of $\tau \in \mathbb{T}$ such that E is invariant under translation by τ is a subgroup of $\mathbb{T}$. Hence prove that if E is a measurable set on $\mathbb{T}$ and E is invariant under translation by infinitely many $\tau \in \mathbb{T}$, then either E or its complement has measure zero.

Hint: A set E of positive measure has points of density, that is, points τ such that $(2\varepsilon)^{-1}|E\cap(\tau-\varepsilon,\tau+\varepsilon)| \to 1$ as $\varepsilon \to 0$. ($|E_0|$ denotes the Lebesgue measure of E_0.)

13. If E and F are subsets of $\mathbb{T}$, we write

$$E + F = \{t + \tau : t \in E, \quad \tau \in F\}$$

and call $E + F$ the algebraic sum of E and F. Similarly we define the sum of any finite number of sets. A set E is called *a basis* for $\mathbb{T}$if there exists an integer N such that $E + E + \cdots + E$ (N times) is $\mathbb{T}$. Prove that every set E of positive measure on $\mathbb{T}$ is a basis.

Hint: Prove that if E contains an interval it is a basis. Using points of density prove that if E has positive measure then $E + E$ contains intervals.

14. Show that measurable proper subgroups of $\mathbb{T}$ have measure zero.

15. Show that measurable homomorphisms of $\mathbb{T}$ into C^* map it into T^*.

16. Let f be a measurable homomorphism of $\mathbb{T}$ into T^*. Show that for all values of n, except possibly one value, $\hat{f}(n) = O$.

2 SUMMABILITY IN NORM AND HOMOGENEOUS BANACH SPACES ON $\mathbb{T}$

2.1 We have defined the Fourier series of a function $f \in L^1(\mathbb{T})$ as a certain (formal) trigonometric series. The reader may wonder what is the point in the introduction of such formal series. After all, there is no more information in the (formal) expression $\sum_{-\infty}^{\infty} \hat{f}(n)e^{int}$ than there is in the simpler one $\{\hat{f}(n)\}_{n=-\infty}^{\infty}$ or the even simpler $\hat{f}$ with the understanding that the function $\hat{f}$ is defined on the integers. As we shall see, both expressions, $\sum \hat{f}(n)e^{int}$ and f, have their advantages;